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On a non-Abelian Hirota–Miwa equation

Jonathan J C Nimmo[1]

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan

E-mail: j.nimmo@maths.gla.ac.uk

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Abstract

A generalization of the Hirota–Miwa equation to an abstract non-Abelian associative algebra is considered. This system is integrable in the sense that it arises as compatibility condition for a linear system and has solutions constructed by means of the application of an arbitrary number of Darboux transformations. These solutions are in general expressed in terms of quasideterminants.

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1. Introduction

Perhaps the most important and widely studied three-dimensional discrete integrable system is the Hirota–Miwa (or discrete KP (Kadomtsev–Petviashvili)) equation

$$\begin{aligned}A_{23}\tau(n_1+1)\tau(n_2+1,n_3+1)+A_{13}\tau(n_2+1)\tau(n_1+1,n_3+1)\\+A_{12}\tau(n_3+1)\tau(n_1+1,n_2+1)=0,\end{aligned} \tag{1}$$

where $\tau=\tau(n_1,n_2,n_3)$ and A_{ij} are arbitrary constants. In the above equation the dependence on unincremented variables is omitted. It was discovered by Hirota [1] as a fully discrete analogue of the two-dimensional Toda equation and later Miwa [2] showed that it was intimately related to the KP (Kadomtsev–Petviashvili) hierarchy. One may regard (1) as a master equation which by taking careful continuum limits gives many well-known semi-discrete and continuous integrable systems. This means that such things as Bäcklund/Darboux transformations, exact solutions, Lax pair, etc for these systems may be obtained by studying the corresponding things for (1). Another important aspect of this that we wish to emphasize is that the simplicity and symmetry of (1), which allow for simple calculations, is broken when one takes continuum limits; simple formulae become complicated. Because of this, there is a good reason to develop as much theory as possible for the fully discrete system and

[1] Permanent address: Department of Mathematics, University of Glasgow, Glasgow G12 8QQ, UK.

then carry these results to the continuum limits. This is a prime motivation for studying the generalization of the system we introduce in this paper.

It is known that the Hirota–Miwa equation has solutions expressed in terms of casoratian determinants which may be constructed by the direct method [3] or by means of Darboux transformations [4]. Considering the details of how these solutions are obtained, it is clear that one may generalize the system in two ways while still maintaining a similar, but much larger, class of solutions. First, one may obtain a generalized system having solutions in a matrix, or more generally a non-Abelian associative, algebra. Also, the lattice on which the system lives, which for (1) is defined in terms of shift operators, may be defined more generally in terms of any commuting operators which act homomorphically on the solutions. By making this abstraction, we will see how we are able to integrate the definition of the system and its Darboux transformations in one uniform expression.

The Hirota–Miwa equation arises as the compatibility condition of either of the two linear systems

$$\phi(n_i+1)-\phi(n_j+1)=\frac{1}{A_{ij}}\frac{\tau(n_i+1)\tau(n_j+1)}{\tau\tau(n_i+1,n_j+1)}\phi(n_i+1,n_j+1) \tag{2}$$

for $1\leqslant i<j\leqslant 3$ or its formal adjoint

$$\phi(n_j+1)-\phi(n_i+1)=\frac{1}{A_{ij}}\frac{\tau(n_i+1)\tau(n_j+1)}{\tau\tau(n_i+1,n_j+1)}\phi. \tag{3}$$

The adjoint system, being first order rather than second order, is simpler to use and it is this that we use to obtain the generalization.

This paper is organized as follows. In section 2, we introduce an infinite-dimensional generalization of (3) in an associative algebra and consider its compatibility conditions. The result is a three-dimensional subsystem which we refer to as a non-Abelian Hirota–Miwa equation. This system is integrable in the sense that it arises as compatibility condition for a linear system and has solutions constructed by means of the application of an arbitrary number of Darboux transformations. In section 3, it is shown that Darboux transformations for this system are found as realizations of the remaining homomorphisms acting on the subsystem. In section 4, we present explicit formulae for solutions of the system obtained by the application of an arbitrary number of Darboux transformations to a vacuum solution. The proofs of these formulae are given in the appendix. In the general case, in which the algebra is not a matrix algebra, determinants are of course not defined and in this case the solutions are expressed in terms of quasideterminants (see [5] and the references therein). However, when we specialize to a matrix system, we describe the solutions one obtains expressed as ratios of determinants.

2. Linear system and its compatibility conditions

Let $\mathcal{A}$ be an associative algebra, over a field F, with multiplicative identity 1. In general, of course, $\mathcal{A}$ is not commutative. Let $\mathcal{A}^\times$ denote the subgroup of elements $\theta\in\mathcal{A}$ having a (left- and right-) multiplicative inverse $\theta^{-1}\in\mathcal{A}$. In the most obvious situation we have in mind, $\mathcal{A}$ would be the algebra of $N\times N$ matrices and $\mathcal{A}^\times$ the subgroup of invertible matrices.

A *homomorphism* is a mapping $s:\mathcal{A}\to\mathcal{A}$ satisfying

$$s(a+b)=s(a)+s(b),\qquad s(ab)=s(a)s(b). \tag{4}$$

Let s_k, $k\in\mathbb{N}$, be a sequence of commuting homomorphisms. The compositions of such mappings $s_{ij\cdots k}=s_i\circ s_j\circ\cdots\circ s_k$ are symmetric in their indices and are themselves homomorphisms. We will also use subscript notation for the action of these homomorphisms;

$A_{,k}$ for $s_k(A)$ and $A_{,ij\cdots k}$ for $s_{ij\cdots k}(A)$. The prototype for the homomorphisms are shift operators; if $a = a(n_1, n_2, \ldots)$ then s_i defined by $s_i(a) = a|_{n_i \to n_i+1}$ are such a sequence of commuting homomorphisms.

A *constant* $\alpha \in \mathcal{A}$ is such that $s_i(\alpha) = \alpha$ for all i. Also, we let $\mathcal{D}$ denote a commutative subalgebra of $\mathcal{A}$ and $\mathcal{D}^\times$ its invertible elements. In the case that $\mathcal{A}$ is a matrix algebra, then $\mathcal{D}$ consists of the diagonal matrices and $\mathcal{D}^\times$ the diagonal matrices with non-zero entries on the diagonal.

In order to explain the main ideas without getting submerged in too much detail, we will not describe the most general situation immediately. Instead, we will consider the simplest situation in which all of the main ideas are evident and then give a more general form later. In the case that the homomorphisms are shift operators, this more general system is related to the one we study in detail through a gauge transformation.

We first consider the system of linear equations

$$\phi_{,i} - \phi_{,j} + U_{ij}\phi = 0, \qquad i, j \in \mathbb{N}, \tag{5}$$

for $\phi \in \mathcal{A}$, where the potentials $U_{ij} \in \mathcal{A}^\times$ (for $i \neq j$) and $U_{ii} = 0$ and is modelled on the adjoint linear problem for the Abelian Hirota–Miwa equation (3). This linear system is compatible if and only if

$$U_{ij} + U_{jk} + U_{ki} = 0, \tag{6}$$

$$U_{ij,k} + U_{jk,i} + U_{ki,j} = 0, \tag{7}$$

$$U_{ij,k}U_{ki} = U_{ki,j}U_{ij}, \tag{8}$$

for each $i, j, k \in \mathbb{N}$. The first set of these conditions expresses the fact that the system is algebraically overdetermined whereas the other two arise from the requirement that all homomorphisms s_i commute. Note, in particular, that (6) includes the condition that the potentials are skew symmetric in their indices; $U_{ii} = 0$ and $U_{ji} = -U_{ij}$ for all $i, j \in \mathbb{N}$.

Apart from these symmetry conditions, for each distinct triple $i, j, k \in \mathbb{N}$ there are in fact *five* relations between the three potentials U_{ij}, U_{jk} and U_{ki} contained within (6)–(8). This is because (8) is not symmetric under permutations of i, j, k and gives three conditions

$$U_{ij,k}U_{ki} = U_{ki,j}U_{ij}, \qquad U_{ij,k}U_{jk} = U_{jk,i}U_{ij}, \qquad U_{jk,i}U_{ki} = U_{ki,j}U_{jk}. \tag{9}$$

However, by considering the sum of the first two equations in (9) and making use of (6) one may deduce (7). For this reason, we may eliminate condition (7) and obtain the compatibility conditions in final form

$$U_{ij} + U_{jk} + U_{ki} = 0, \tag{10}$$

$$U_{ij,k}U_{ki} = U_{ki,j}U_{ij}. \tag{11}$$

Next, we introduce $X_i \in \mathcal{A}^\times$ and $\alpha_{ii} = 0$ and $\alpha_{ij} \in \mathcal{D}^\times$ (for $i \neq j$) satisfying (11). Then, we make the ansatz

$$U_{ij} = X_{i,j}\alpha_{ij}X_i^{-1} = -X_{j,i}\alpha_{ji}X_j^{-1}, \tag{12}$$

which satisfies (11) identically. So we see that the compatibility condition can be reduced to the purely algebraic consistency condition (10). It is through the nonconstancy of α_{ij} that inhomogeneity may enter the system, but here we will assume that they are constant.

In the Abelian case, we can introduce a single τ-function and make the further ansatz $X_i = \tau_{,i}/\tau$. Then, for each distinct triple i, j, k, the four equations in (10) give only one nontrivial condition

$$\alpha_{ij}\tau_{,ij}\tau_{,k} + \alpha_{jk}\tau_{,jk}\tau_{,i} + \alpha_{ki}\tau_{,ki}\tau_{,j} = 0 \tag{13}$$

and the requirement that the α_{pq} are skew symmetric in their indices. This is simply a form of the Hirota–Miwa equation if one takes $\tau = \tau(n_1, n_2, \ldots)$ and uses the homomorphisms as $\tau_{,i} = \tau|_{n_i \to n_i+1}$. For this reason, we refer to the system under consideration as a *non-Abelian Hirota–Miwa equation* [1, 2].

Returning to, and summarizing, the non-Abelian case, we have proved the following result.

Theorem 1. *The linear system*

$$\phi_{,i} - \phi_{,j} + U_{ij}\phi = 0, \tag{14}$$

where $U_{ij} = X_{i,j}\alpha_{ij}X_i^{-1}$, is compatible for all ϕ if and only if

$$U_{ij} + U_{jk} + U_{ki} = 0. \tag{15}$$

For each distinct triple i, j, k, this is a system of four nonlinear equations in the three unknowns $X_i, X_j, X_k \in \mathcal{A}^\times$.

If we assume that the homomorphisms s_1, s_2, s_3 are shift operators defined on a three-dimensional lattice defined by variables n_1, n_2, n_3, then we may use a gauge transformation

$$\phi \to \prod_{k\geqslant 1} \alpha_k^{n_k}\phi, \qquad X_i \to \prod_{k\geqslant 1} \alpha_k^{n_k} X_i, \tag{16}$$

where α_k are commuting, constant and invertible, to generalize the linear and nonlinear systems.

Theorem 2. *The linear system*

$$\alpha_i\phi_{,i} - \alpha_j\phi_{,j} + U_{ij}\phi = 0, \tag{17}$$

where $U_{ij} = \alpha_j X_{i,j}\alpha_{ij}X_i^{-1}$, is compatible for all ϕ if and only if

$$U_{ij} + U_{jk} + U_{ki} = 0. \tag{18}$$

Although this result is being presented as a simple corollary of theorem 1, it is in fact also valid for *any* homomorphisms. But, since in other cases there may be no gauge transformation, it must be proved directly in those cases, mimicking the earlier proof. Throughout this paper, we will only consider the simplest version of the system given in theorem 1 but note that all results can readily be generalized to apply for the system described in theorem 2.

3. Darboux transformations as homomorphisms

Let us now fix on one such triple, the obvious choice being 1, 2, 3. The nonlinear system referred to in theorem 1 is given explicitly by

$$X_{1,2}\alpha_{12}X_1^{-1} + X_{2,3}\alpha_{23}X_2^{-1} + X_{3,1}\alpha_{31}X_3^{-1} = 0, \tag{19}$$

$$X_{1,2}\alpha_{12}X_1^{-1} + X_{2,1}\alpha_{21}X_2^{-1} = X_{2,3}\alpha_{23}X_2^{-1} + X_{3,2}\alpha_{32}X_3^{-1} = X_{3,1}\alpha_{31}X_3^{-1} + X_{1,3}\alpha_{13}X_1^{-1} = 0. \tag{20}$$

The remaining homomorphisms, which we write as $d_k = s_{k+3}$ for $k \in \mathbb{N}$, can act on this system and because they are homomorphisms, and so preserve the structure of both the nonlinear equations and the linear system from which they arise, they may be thought of as abstract Darboux/Bäcklund transformations, giving new eigenfunctions $d_k(\phi)$ and new solutions $d_k(X_i)$ in terms of known ones. In this section, we will obtain some concrete expressions for such transformations and thereby construct a family of exact solutions.

First, we fix a particular vacuum solution X_1, X_2, X_3; for example, if $\alpha_{ij}+\alpha_{jk}+\alpha_{ki}=0$ for $i, j, k \in \{1, 2, 3\}$, then we may choose the trivial solution $X_1 = X_2 = X_3 = 1$. Also, suppose that we may solve the associated linear problem (17) (for $i, j \in \{1, 2, 3\}$) with this vacuum potential. Let $\mathcal{E}$ and $\mathcal{E}^\times$ denote the sets of eigenfunctions and invertible eigenfunctions, respectively. We will express each d_k in terms of an eigenfunction $\theta_k \in \mathcal{E}^\times$. There will be further conditions on θ_k introduced later relating to the independence of these eigenfunctions and to nonsingularity.

To begin this construction, suppose that $d_k(\theta_k) = \theta_{k,k+3} = 0$. From (17), for $i = k+3$ with $\phi = \theta_k$, we obtain the higher potentials

$$U_{k+3j} = \theta_{k,j}\theta_k^{-1}, \qquad k \in \mathbb{N}, \qquad j = 1, 2, 3.$$

From (12), we also have two other expressions for these potentials:

$$U_{k+3j} = X_{k+3,j}\alpha_{k+3j}X_{k+3}^{-1} = -d_k(X_j)\alpha_{jk+3}X_j^{-1}.$$

Comparing these expressions, it is clear that we may choose $X_{k+3} = \theta_k$ and $\alpha_{k+3j} = \alpha_{jk+3} = 1$ and then

$$d_k(X_j) = -\theta_{k,j}\theta_k^{-1}X_j, \qquad k \in \mathbb{N}, \quad j = 1, 2, 3. \tag{21}$$

Also, from (14) for $i = k+3$ it follows that

$$d_k(\phi) = \phi_{,j} - \theta_{k,j}\theta_k^{-1}\phi, \qquad k \in \mathbb{N}, \quad j = 1, 2 \text{ or } 3. \tag{22}$$

In (22), the construction naturally gives three different expressions for $d_k(\phi)$. Although it is unnecessary to do so, one may also check directly that since both ϕ and θ_k are eigenfunctions for the vacuum potential, all are equal. This is the familiar expression for the discrete Darboux transformation for the Hirota–Miwa equation (see for example [4, 6]) extended to the non-Abelian case. It is noteworthy that it arises in an entirely natural way as a manifestation of the system itself and not as some externally imposed transformation.

Finally, for completeness we note that all of the higher potentials are determined in terms of the eigenfunctions θ_k. Using (12) with $X_{k+3} = \theta_k$ and $\alpha_{ij} = 1$, we obtain U_{ij} for $i > 3$ and/or $j > 3$.

So far we only have explicit formulae for d_k when it acts on vacuum eigenfunctions, (22), and on the vacuum potentials X_i, (21). This shows, in particular, that if (X_1, X_2, X_3) is a known solution of (19)–(20) and θ_k a corresponding eigenfunction, then $\left(-\theta_{k,1}\theta_k^{-1}X_1, -\theta_{k,2}\theta_k^{-1}X_2, -\theta_{k,3}\theta_k^{-1}X_3\right)$ is also a solution. However, d_k is by definition one of the sequences of commuting homomorphisms s_i and we will make use of the properties of such operators in the next section to obtain other explicit formulae corresponding to repeated application of Darboux transformations and hence obtain a rich family of solutions.

4. Iteration of Darboux transformations

In this section, we will describe some properties of matrices over the noncommutative algebra $\mathcal{A}$ and how they may be used to find expressions for solution obtained by the application of multiple Darboux transformations. The topic of matrices of noncommuting objects has a long history and has been studied in depth in recent years by Gelfand, Retakh and co-workers who developed the theory of *quasideterminants* and this has recently been extensively reviewed in [5]. Here we do not attempt to use the general theory but simply mention a few basic formulae. We also observe that this theory has been used before in the context of integrable systems [7, 8] but for continuous rather than discrete systems.

4.1. Matrix inversion over $\mathcal{A}$ and the Schur complement

Let $a, b, c, d \in \mathcal{A}$ and consider the matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}_{2\times 2}.$$

If a is invertible then the *Schur complement of a in M* is defined to be $(M|a) = d - ca^{-1}b$ (see [9]). This is also the simplest type of quasideterminant [5]. If $(M|a)$ is invertible then so is M and

$$M^{-1} = \begin{bmatrix} a^{-1} + a^{-1}b(M|a)^{-1}ca^{-1} & -a^{-1}b(M|a)^{-1} \\ -(M|a)^{-1}ca^{-1} & (M|a)^{-1} \end{bmatrix}. \tag{23}$$

In fact, these results remain valid if we replace elements a, b, c, d with matrices A, B, C, D over $\mathcal{A}$ of appropriate dimensions. Assuming that A and $(M|A) := D - CA^{-1}B$ are invertible then

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(M|A)^{-1}CA^{-1} & -A^{-1}B(M|A)^{-1} \\ -(M|A)^{-1}CA^{-1} & (M|A)^{-1} \end{bmatrix}. \tag{24}$$

Note also that if A is invertible then

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & A & B \\ I & C & D \end{bmatrix}^{-1} = \begin{bmatrix} -(M|A) & -CA^{-1} & I \\ -A^{-1}B & A^{-1} & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{25}$$

where I and $\mathbf{0}$ denote identity matrices and zero matrices of appropriate size. This means that one may obtain the Schur complement by projecting onto the (1, 1) block in the inverse of the bordered matrix shown in (25). Explicitly we have

$$(M|A) = D - CA^{-1}B = -[I \quad \mathbf{0} \quad \mathbf{0}] \begin{bmatrix} \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & A & B \\ I & C & D \end{bmatrix}^{-1} \begin{bmatrix} I \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & A & B \\ I & C & D \end{bmatrix}^{p}. \tag{26}$$

More generally, using (26), it is easy to prove the following lemma.

Lemma 1. *Let M and A be as mentioned above and let M_1 be a matrix such that $(M|A) = M_1^p$. Then $M_2 = PM_1Q$, where P and Q are invertible and*

$$P = \begin{bmatrix} I & \cdot & \cdot \\ \mathbf{0} & \cdot & \cdot \\ \mathbf{0} & \cdot & \cdot \end{bmatrix}, \qquad Q = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix},$$

is such that $(M|A) = M_2^p$ also.

In particular, in proving some of the results on iterated Darboux transformations the following alternative formulae may be used:

$$(M|A) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & A & B \\ I & C & D \end{bmatrix}^{p} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & I \\ I & C & D \\ \mathbf{0} & A & B \end{bmatrix}^{p} = \begin{bmatrix} \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & B & A \\ I & D & C \end{bmatrix}^{p} = \begin{bmatrix} \mathbf{0} & I & \mathbf{0} \\ I & D & C \\ \mathbf{0} & B & A \end{bmatrix}^{p}. \tag{27}$$

Finally, in the case that $\mathcal{A}$ is a matrix algebra, we may use expression (26) to express the entries in the matrix $(M|A)$ as ratios of determinants

$$(M|A)_{ij} = \frac{\begin{vmatrix} A & c_i(B) \\ r_j(C) & D_{ij} \end{vmatrix}}{|A|}, \tag{28}$$

where $c_i(B)$ denotes the ith column of B and $r_j(C)$ the jth row of C.

4.2. Reformulation of Darboux transformations

The first use of these formulae is made in re-expressing the action (21), (22) of a single Darboux transformation in the notation introduced in (26):

$$d_k(\phi) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \theta_k & \phi \\ 1 & \theta_{k,i} & \phi_{,i} \end{bmatrix}^p \tag{29}$$

and

$$d_k(X_i) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \theta_k & 1 \\ 1 & \theta_{k,i} & 0 \end{bmatrix}^p X_i, \tag{30}$$

for $k \in \mathbb{N}$ and $i = 1, 2$ or 3.

Recall that d_k are at present only defined on vacuum eigenfunctions and potentials. So, when calculating $d_k(\phi_{,i})$, for example, we must first use the fact that d_k and s_i commute and write it as $(d_k(\phi))_{,i}$. Also, if θ and $d_k(\theta)$ are invertible, we may use the homomorphic nature of d_k to obtain $d_k(\theta^{-1}) = d_k(\theta)^{-1}$. Proceeding to consider the second degree Darboux transformations $d_{kl} := d_l \circ d_k = d_k \circ d_l$, assuming that $d_k(\theta_l)$ is nonsingular, we have

$$\begin{aligned} d_{kl}(\phi) &= d_k\big(\phi_{,i} - \theta_{l,i}\theta_l^{-1}\phi\big) \\ &= (d_k(\phi))_{,i} - (d_k(\theta_l))_{,i} d_k(\theta_l)^{-1} d_k(\phi) \\ &= \phi_{,ii} - \theta_{k,ii}\theta_{k,i}^{-1}\phi_{,i} - \big(\theta_{l,ii} - \theta_{k,ii}\theta_{k,i}^{-1}\theta_{l,i}\big)\big(\theta_{l,i} - \theta_{k,i}\theta_k^{-1}\theta_l\big)^{-1}\big(\phi_{,i} - \theta_{k,i}\theta_k^{-1}\phi\big), \end{aligned}$$

for any $i \in \{1, 2, 3\}$. It is not at all obvious from this expression that it is invariant under interchange of k and l, as we know by construction that it should be. However, by straightforward manipulation, this expression can be rewritten using (23) in the more obviously symmetric form

$$d_{kl}(\phi) = s_{ii}(\phi) - [s_{ii}(\theta_k) \quad s_{ii}(\theta_l)] \begin{bmatrix} \theta_k & \theta_l \\ s_i(\theta_k) & s_i(\theta_l) \end{bmatrix}^{-1} \begin{bmatrix} \phi \\ s_i(\phi) \end{bmatrix}. \tag{31}$$

Then, using (27), this can also be written as

$$d_{kl}(\phi) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \theta_k & \theta_l & \phi \\ 0 & s_i(\theta_k) & s_i(\theta_l) & s_i(\phi) \\ 1 & s_{ii}(\theta_k) & s_{ii}(\theta_l) & s_{ii}(\phi) \end{bmatrix}^p. \tag{32}$$

By interchanging the middle columns and using the invariance described in lemma 1, one confirms that the Darboux transformations d_k indeed commute. In a similar way,

$$\begin{aligned} d_{kl}(X_i) &= d_k(-\theta_{l,i}\theta_l^{-1}X_i) \\ &= -(d_k(\theta_l))_{,i} d_k(\theta_l)^{-1} d_k(X_i) \\ &= \big(\theta_{l,ii} - \theta_{k,ii}\theta_{k,i}^{-1}\theta_{l,i}\big)\big(\theta_{l,i} - \theta_{k,i}\theta_k^{-1}\theta_l\big)^{-1}\theta_{k,i}\theta_k^{-1}X_i, \end{aligned}$$

which may be rewritten as

$$d_{kl}(X_i) = [\theta_{k,ii} \quad \theta_{l,ii}] \begin{bmatrix} \theta_k & \theta_l \\ \theta_{k,i} & \theta_{l,i} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} X_i, \tag{33}$$

which, using (26), is

$$d_{kl}(X_i) = -\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \theta_k & \theta_l & 1 \\ 0 & s_i(\theta_k) & s_i(\theta_l) & 0 \\ 1 & s_{ii}(\theta_k) & s_{ii}(\theta_l) & 0 \end{bmatrix}^p X_i. \tag{34}$$

The general results for the composition of an arbitrary number of Darboux transformations $d_1, \ldots, d_n$ are summarized in the following theorem. The explicit formulae are proved in the appendix.

Theorem 3. *Let $X_i, i = 1, 2, 3$, satisfy (19)–(20) and let θ_k, $k = 1, \ldots, n$, be the corresponding eigenfunctions, solutions of (17), from which commuting homomorphic Darboux transformations d_k may be defined by (21), (22). The composition of these $d_{1,\ldots,n} = d_1 \circ \cdots \circ d_n$ is also a Darboux transformation and so*

$$d_{1,\ldots,n}(X_i) = (-1)^{n-1} \begin{bmatrix} 0 & \mathbf{0} & 1 \\ 0 & \boldsymbol{\theta} & 1 \\ 0 & s_i(\boldsymbol{\theta}) & 0 \\ \vdots & \vdots & \vdots \\ 0 & s_i^{n-1}(\boldsymbol{\theta}) & 0 \\ 1 & s_i^n(\boldsymbol{\theta}) & 0 \end{bmatrix}^p X_i, \tag{35}$$

$i = 1, 2, 3$, where $\boldsymbol{\theta} = [\theta_1 \cdots \theta_n]$, also satisfy (19)–(20). The corresponding eigenfunctions are

$$d_{1,\ldots,n}(\phi) = \begin{bmatrix} 0 & \mathbf{0} & 1 \\ 0 & \boldsymbol{\theta} & \phi \\ 0 & s_i(\boldsymbol{\theta}) & s_i(\phi) \\ \vdots & \vdots & \vdots \\ 0 & s_i^{n-1}(\boldsymbol{\theta}) & s_i^{n-1}(\phi) \\ 1 & s_i^n(\boldsymbol{\theta}) & s_i^n(\phi) \end{bmatrix}^p. \tag{36}$$

Further, one may also calculate the effect of the Darboux transformations on X_i^{-1}:

$$d_{1,\ldots,n}\left(X_i^{-1}\right) = (-1)^{n-1} X_i^{-1} \begin{bmatrix} 0 & \mathbf{0} & 1 \\ 1 & \boldsymbol{\theta} & 0 \\ 0 & s_i(\boldsymbol{\theta}) & 0 \\ \vdots & \vdots & \vdots \\ 0 & s_i^{n-1}(\boldsymbol{\theta}) & 0 \\ 0 & s_i^n(\boldsymbol{\theta}) & 1 \end{bmatrix}^p. \tag{37}$$

4.3. Specialization to a matrix algebra

Finally, we give some details of some results to be presented in more detail in a subsequent paper [10] on what more can be done when one takes the simplest concrete choice of $\mathcal{A}$ as the algebra of $N \times N$ complex matrices $M_{N\times N}(\mathbb{C})$ and s_1, s_2, s_3 are shift operators with respect to variables n_1, n_2, n_3, respectively.

For simplicity, choose α_{ij} to be scalar and skew symmetric in indicies i, j so that we may choose the simple vacuum solution $X_i = 1$ of (18). One may make the ansätze $X_i = G_i/F$

and $X_i^{-1} = H_i/F_{,i}$, where G_i, H_i are matrix valued and F is a scalar function of n_1, n_2, n_3. Then, the nonlinear system (18) becomes

$$\alpha_{12}G_{1,2}H_1F_{,3} + \alpha_{23}G_{2,3}H_2F_{,1} + \alpha_{31}G_{3,1}H_3F_{,2} = 0, \tag{38}$$

$$G_{1,2}H_1 - G_{2,1}H_2 = G_{2,3}H_2 - G_{3,2}H_3 = G_{3,1}H_3 - G_{1,3}H_1 = 0, \tag{39}$$

and in addition

$$G_1H_1 = FF_{,1}, \qquad G_2H_2 = FF_{,2}, \qquad G_3H_3 = FF_{,3}, \tag{40}$$

which might be called *matrix Hirota–Miwa equations*.

This system has solutions obtained from (35) and (37) in which each θ_k is an invertible $N \times N$ matrix solution of the linear difference equations (17), and so $\boldsymbol{\theta}$ is an $N \times nN$ matrix, and 1 is the $N \times N$ identity matrix. Then, using (28), one obtains expressions for F and the (i, j)th entries in the matrices G_k and H_k as casoratian-like determinants:

$$F = \begin{vmatrix} \boldsymbol{\theta} \\ s_i(\boldsymbol{\theta}) \\ \vdots \\ s_i^{n-1}(\boldsymbol{\theta}) \end{vmatrix},$$

where $i = 1, 2$ or 3, and

$$(G_k)_{ij} = \begin{vmatrix} \boldsymbol{\theta}_{\hat{j}} \\ s_k(\boldsymbol{\theta}) \\ \vdots \\ s_k^{n-1}(\boldsymbol{\theta}) \\ s_k^{n-1}(\boldsymbol{\theta}_i) \end{vmatrix}, \qquad (H_k)_{ij} = \begin{vmatrix} \boldsymbol{\theta}_i \\ s_k(\boldsymbol{\theta}) \\ \vdots \\ s_k^{n-1}(\boldsymbol{\theta}) \\ s_k^{n-1}(\boldsymbol{\theta}_{\hat{j}}) \end{vmatrix},$$

where $\boldsymbol{\theta}_i$ denotes the ith row of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_{\hat{j}}$ the matrix obtained by deleting the jth row from $\boldsymbol{\theta}$.

This approach allows for a direct verification of the solution that is rather reminiscent of the bilinear approach to the self-dual Yang–Mills equations given in [11]. This will be reported on in full in [10].

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Appendix. Proofs of iteration formulae

Here, we will give proofs of the general formulae expressing the action of n of Darboux transformations on vacuum eigenfunctions and potentials. All proofs are by induction on n.

Notation. In the proofs below we will employ the following shorthand notation. For any given $n \in \mathbb{N}$ and $i = 1, 2$ or 3, and $k = n$ or $n + 1$,

$$\boldsymbol{\theta} = [\theta_1 \ \cdots \ \theta_n], \qquad \boldsymbol{\theta}^k = big[s_i^k(\theta_1) \ \cdots \ s_i^k(\theta_n)\big],$$
$$\phi^k = s_i^k(\phi), \qquad \theta_{n+1}^k = s_i^k(\theta_{n+1}),$$

$$\Phi = \begin{bmatrix} \phi \\ s_i(\phi) \\ \vdots \\ s_i^{n-1}(\phi) \end{bmatrix}, \qquad \Theta_{n+1} = \begin{bmatrix} \theta_{n+1} \\ s_i(\theta_{n+1}) \\ \vdots \\ s_i^{n-1}(\theta_{n+1}) \end{bmatrix}$$

and

$$\Theta = \begin{bmatrix} \theta_1 & \cdots & \theta_n \\ s_i(\theta_1) & \cdots & s_i(\theta_n) \\ \vdots & \vdots & \\ s_i^{n-1}(\theta_1) & \cdots & s_i^{n-1}(\theta_n) \end{bmatrix}.$$

We also define the $n \times 1$ projection matrices

$$P = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Proof of (36). The result for $n = 1$ was given in (29).

Now assume that the result is true for a given $n \in \mathbb{N}$. Then, using (26), we have

$$d_{1,\ldots,n}(\phi) = \phi^n - \boldsymbol{\theta}^n \Theta^{-1} \Phi, \qquad d_{1,\ldots,n}(\theta_{n+1}) = \theta_{n+1}^n - \boldsymbol{\theta}^n \Theta^{-1} \Theta_{n+1}. \tag{A.1}$$

Now consider the matrix M_{n+1} used in the expression for $d_{1,\ldots,n+1}(\phi)$, partitioned in two distinct way,

$$M_{n+1} = \begin{bmatrix} \text{[|..||]}0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & \theta_1 & \cdots & \theta_n & \theta_{n+1} & \phi \\ 0 & s_i(\theta_1) & \cdots & s_i(\theta_n) & s_i(\theta_{n+1}) & s_i(\phi) \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & s_i^{n-1}(\theta_1) & \cdots & s_i^{n-1}(\theta_n) & s_i^{n-1}(\theta_{n+1}) & s_i^{n-1}(\phi) \\ 0 & s_i^{n}(\theta_1) & \cdots & s_i^{n}(\theta_n) & s_i^{n}(\theta_{n+1}) & s_i^{n}(\phi) \\ 1 & s_i^{n+1}(\theta_1) & \cdots & s_i^{n+1}(\theta_n) & s_i^{n+1}(\theta_{n+1}) & s_i^{n+1}(\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \Theta & \Theta_{n+1} & \Phi \\ 0 & \boldsymbol{\theta}^n & \theta_{n+1}^n & \phi^n \\ 1 & \boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1} & \phi^{n+1} \end{bmatrix}$$

and

$$\widetilde{M}_{n+1} = \begin{bmatrix} \text{[|..||]}0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & \theta_1 & \cdots & \theta_n & \theta_{n+1} & \phi \\ 0 & s_i(\theta_1) & \cdots & s_i(\theta_n) & s_i(\theta_{n+1}) & s_i(\phi) \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & s_i^{n}(\theta_1) & \cdots & s_i^{n}(\theta_n) & s_i^{n}(\theta_{n+1}) & s_i^{n}(\phi) \\ 1 & s_i^{n+1}(\theta_1) & \cdots & s_i^{n+1}(\theta_n) & s_i^{n+1}(\theta_{n+1}) & s_i^{n+1}(\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathbf{0} & 0 & 1 \\ 0 & \boldsymbol{\theta} & \theta_{n+1} & \phi \\ \mathbf{0} & s_i(\Theta) & s_i(\Theta_{n+1}) & s_i(\Phi) \\ 1 & \boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1} & \phi^{n+1} \end{bmatrix}.$$

Of course $M_{n+1} = \widetilde{M}_{n+1}$; they are simply different partitionings of the same matrix.

To prove the inductive step, we need to show that M^p_{n+1} equals

$$d_{1,\ldots,n+1}(\phi) = d_{1,\ldots,n}(d_{n+1}(\phi)) = s_i(d_{1,\ldots,n}(\phi)) - s_i(d_{1,\ldots,n}(\theta_{n+1}))d_{1,\ldots,n}(\theta_{n+1})^{-1}d_{1,\ldots,n}(\phi). \tag{A.2}$$

To achieve this, we will need a different, less obvious expression obtained using both M_{n+1} and $\widetilde{M}_{n+1}$. First note, considering the central blocks of M_{n+1} and $\widetilde{M}_{n+1}$, that we have

$$\begin{bmatrix} \Theta & \Theta_{n+1} \\ \boldsymbol{\theta}^n & \theta^n_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & 1 \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} s_i(\Theta) & s_i(\Theta_{n+1}) \\ \boldsymbol{\theta} & \theta_{n+1} \end{bmatrix}, \qquad \begin{bmatrix} \Phi \\ \phi^n \end{bmatrix} = \begin{bmatrix} \mathbf{0} & 1 \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} s_i(\Phi) \\ \phi \end{bmatrix}$$

and so

$$\begin{bmatrix} \Theta & \Theta_{n+1} \\ \boldsymbol{\theta}^n & \theta^n_{n+1} \end{bmatrix}^{-1} \begin{bmatrix} \Phi \\ \phi^n \end{bmatrix} = \begin{bmatrix} s_i(\Theta) & s_i(\Theta_{n+1}) \\ \boldsymbol{\theta} & \theta_{n+1} \end{bmatrix}^{-1} \begin{bmatrix} s_i(\Phi) \\ \phi \end{bmatrix}.$$

Using (24) and (A.1), this gives

$$\begin{bmatrix} \Theta^{-1}\Phi - \Theta^{-1}\Theta_{n+1}d_{1,\ldots,n}(\theta_{n+1})^{-1}d_{1,\ldots,n}(\phi) \\ d_{1,\ldots,n}(\theta_{n+1})^{-1}d_{1,\ldots,n}(\phi) \end{bmatrix}$$
$$= \begin{bmatrix} s_i(\Theta)^{-1}s_i(\Phi) - s_i(\Theta)^{-1}s_i(\Theta_{n+1})\widetilde{S}^{-1}(\phi - \boldsymbol{\theta}s_i(\Theta)^{-1}s_i(\Phi)) \\ \widetilde{S}^{-1}(\phi - \boldsymbol{\theta}s_i(\Theta)^{-1}s_i(\Phi)) \end{bmatrix},$$

where $\widetilde{S} := \theta_{n+1} - \boldsymbol{\theta}s_i(\Theta)^{-1}s_i(\Theta_{n+1})$. From this we get

$$\begin{bmatrix} \Theta & \Theta_{n+1} \\ \boldsymbol{\theta}^n & \theta^n_{n+1} \end{bmatrix}^{-1} \begin{bmatrix} \Phi \\ \phi^n \end{bmatrix} = \begin{bmatrix} s_i(\Theta)^{-1}s_i(\Phi) - s_i(\Theta)^{-1}s_i(\Theta_{n+1})d_{1,\ldots,n}(\theta_{n+1})^{-1}d_{1,\ldots,n}(\phi) \\ d_{1,\ldots,n}(\theta_{n+1})^{-1}d_{1,\ldots,n}(\phi) \end{bmatrix}. \tag{A.3}$$

Finally, we may use (27) and (A.3) to calculate

$$\begin{aligned} M^p_{n+1} &= \phi^{n+1} - \begin{bmatrix} \boldsymbol{\theta}^{n+1} & \theta^{n+1}_{n+1} \end{bmatrix} \begin{bmatrix} \Theta & \Theta_{n+1} \\ \boldsymbol{\theta}^n & \theta^n_{n+1} \end{bmatrix}^{-1} \begin{bmatrix} \Phi \\ \phi^n \end{bmatrix} \\ &= s_i(d_{1,\ldots,n}(\phi)) - s_i(d_{1,\ldots,n}(\theta_{n+1}))d_{1,\ldots,n}(\theta_{n+1})^{-1}d_{1,\ldots,n}(\phi). \end{aligned}$$

This establishes (A.2) and the proof is complete. □

Proof of (35). This was proved for $n = 1$ in (30).

Consider

$$M_{n+1} = \begin{bmatrix} \text{[|..||]}0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & \theta_1 & \cdots & \theta_n & \theta_{n+1} & 1 \\ 0 & s_i(\theta_1) & \cdots & s_i(\theta_n) & s_i(\theta_{n+1}) & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & s_i^{n-1}(\theta_1) & \cdots & s_i^{n-1}(\theta_n) & s_i^{n-1}(\theta_{n+1}) & 0 \\ 0 & s_i^n(\theta_1) & \cdots & s_i^n(\theta_n) & s_i^n(\theta_{n+1}) & 0 \\ 1 & s_i^{n+1}(\theta_1) & \cdots & s_i^{n+1}(\theta_n) & s_i^{n+1}(\theta_{n+1}) & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \Theta & \Theta_{n+1} & P \\ 0 & \boldsymbol{\theta}^n & \theta^n_n & 0 \\ 1 & \boldsymbol{\theta}^{n+1} & \theta^{n+1}_{n+1} & 0 \end{bmatrix}$$

and the alternative partitioning

$$\widetilde{M}_{n+1} = \begin{bmatrix} \text{[|..||]}0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & \theta_1 & \cdots & \theta_n & \theta_{n+1} & 1 \\ 0 & s_i(\theta_1) & \cdots & s_i(\theta_n) & s_i(\theta_{n+1}) & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & s_i^n(\theta_1) & \cdots & s_i^n(\theta_n) & s_i^n(\theta_{n+1}) & 0 \\ 1 & s_i^{n+1}(\theta_1) & \cdots & s_i^{n+1}(\theta_n) & s_i^{n+1}(\theta_{n+1}) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{0} & 0 & 1 \\ 0 & \boldsymbol{\theta} & \theta_{n+1} & 1 \\ \mathbf{0} & s_i(\Theta) & s_i(\Theta_{n+1}) & \mathbf{0} \\ 1 & \boldsymbol{\theta}^{n+1} & \theta^{n+1}_{n+1} & 0 \end{bmatrix}.$$

To prove the inductive step, we consider

$$\begin{aligned} d_{1,\dots,n+1}(X_i) &= d_{1,\dots,n}(d_{n+1}(X_i)) = s_i(d_{1,\dots,n}(\theta_{n+1}))d_{1,\dots,n}(\theta_{n+1})^{-1}d_{1,\dots,n}(X_i) \\ &= (-1)^{n-1}s_i(d_{1,\dots,n}(\theta_{n+1}))d_{1,\dots,n}(\theta_{n+1})^{-1}\boldsymbol{\theta}^n\Theta^{-1}PX_i, \end{aligned}$$

and so to complete the proof we must show that

$$M_{n+1}^p = s_i(d_{1,\dots,n}(\theta_{n+1}))d_{1,\dots,n}(\theta_{n+1})^{-1}\boldsymbol{\theta}^n\Theta^{-1}P. \tag{A.4}$$

Using (27) and a combination of the two partitionings of M_{n+1} as was done in the proof of (35), we get

$$\begin{aligned} M_{n+1}^p &= -\begin{bmatrix}\boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1}\end{bmatrix}\begin{bmatrix}\Theta & \Theta_{n+1}\\ \boldsymbol{\theta}^n & \theta_n^n\end{bmatrix}^{-1}\begin{bmatrix}P\\0\end{bmatrix} \\ &= -\begin{bmatrix}\boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1}\end{bmatrix}\begin{bmatrix}s_i(\Theta)^{-1}s_i(\Theta_{n+1})d_{1,\dots,n}(\theta_{n+1})^{-1}\boldsymbol{\theta}^n\Theta^{-1}P\\ -d_{1,\dots,n}(\theta_{n+1})^{-1}\boldsymbol{\theta}^n\Theta^{-1}P\end{bmatrix} \\ &= s_i(d_{1,\dots,n}(\theta_{n+1}))d_{1,\dots,n}(\theta_{n+1})^{-1}\boldsymbol{\theta}^n\Theta^{-1}P, \end{aligned}$$

thus verifying (A.4) and hence (37).

In passing we note that using the partitioning of $\widetilde{M}_{n+1}$ defined above, we get

$$\begin{aligned} \widetilde{M}_{n+1}^p &= \begin{bmatrix} 0 & \mathbf{0} & 0 & 1\\ \mathbf{0} & s_i(\Theta) & s_i(\Theta_{n+1}) & \mathbf{0}\\ 0 & \boldsymbol{\theta} & \theta_{n+1} & 1\\ 1 & \boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1} & 0 \end{bmatrix}^p \\ &= -\begin{bmatrix}\boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1}\end{bmatrix}\begin{bmatrix}s_i(\Theta) & s_i(\Theta_{n+1})\\ \boldsymbol{\theta} & \theta_{n+1}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\1\end{bmatrix} \\ &= -\left(\theta_{n+1}^{n+1} - \boldsymbol{\theta}^{n+1}s_i(\Theta)^{-1}s_i(\Theta_{n+1})\right)\left(\theta_{n+1} - \boldsymbol{\theta}\Theta^{-1}\Theta_{n+1}\right)^{-1}, \end{aligned}$$

and so

$$d_{1,\dots,n+1}(X_i) = (-1)^{n-1}\left(\theta_{n+1}^{n+1} - \boldsymbol{\theta}^{n+1}s_i(\Theta)^{-1}s_i(\Theta_{n+1})\right)\left(\theta_{n+1} - \boldsymbol{\theta}\Theta^{-1}\Theta_{n+1}\right)^{-1}X_i. \tag{A.5}$$

The first factor on the right-hand side is $s_i(d_{1,\dots,n}(\theta_{n+1}))$. □

Proof of (37). For $n = 1$, we have, from (21),

$$d_1(X_i) = -s_i(\theta_1)\theta_1^{-1}X_i,$$

and so, using (27),

$$d_1\left(X_i^{-1}\right) = -X_i^{-1}\theta_1 s_i(\theta_1)^{-1} = X_i^{-1}\begin{bmatrix}0 & 0 & 1\\ 1 & \theta_1 & 0\\ 0 & s_i(\theta_1) & 1\end{bmatrix}^p,$$

as required.

Otherwise, we may use the factorization (A.5) to obtain

$$d_{1,\dots,n+1}\left(X_i^{-1}\right) = (-1)^{n-1}X_i^{-1}\left(\theta_{n+1} - \boldsymbol{\theta}\Theta^{-1}\Theta_{n+1}\right)\left(\theta_{n+1}^{n+1} - \boldsymbol{\theta}^{n+1}s_i(\Theta)^{-1}s_i(\Theta_{n+1})\right)^{-1}.$$

It is straightforward to follow the steps in the proof of (A.5) and show that

$$\begin{aligned} -\left(\theta_{n+1} - \boldsymbol{\theta}\Theta^{-1}\Theta_{n+1}\right)\left(\theta_{n+1}^{n+1} - \boldsymbol{\theta}^{n+1}s_i(\Theta)^{-1}s_i(\Theta_{n+1})\right)^{-1} &= \begin{bmatrix} 0 & \mathbf{0} & 0 & 1\\ \mathbf{0} & s_i(\Theta) & s_i(\Theta_{n+1}) & \mathbf{0}\\ 0 & \boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1} & 1\\ 1 & \boldsymbol{\theta} & \theta_{n+1} & 0 \end{bmatrix}^p \\ &= \begin{bmatrix} 0 & \mathbf{0} & 0 & 1\\ P & \Theta & \Theta_{n+1} & \mathbf{0}\\ 0 & \boldsymbol{\theta}^n & \theta_{n+1}^{n+1} & 0\\ 0 & \boldsymbol{\theta}^{n+1} & \theta_{n+1}^{n+1} & 1 \end{bmatrix}^p, \end{aligned}$$

which completes the inductive step. □

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